Given that $\frac{dy}{dx} = x^2 - xy - 1$ and that $y = 1$ when $x = 0$,

(a) show that for values of x close to 0,

$$y = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 \quad \text{(6 marks)}$$

(b) use the approximation

$$2n\left|\frac{dy}{dx}\right|_n \approx y_{n+1} - y_{n-1}$$

and a step length of 0.2 to find the value of y when $x = 0.4$, giving your final answer correct to 4 decimal places.

(6 marks)

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7. The random variable X has probability density function defined by

$$f(x) = \begin{cases} \lambda(1 - x^2), 0 \le x \le 1 \\ 0, \text{ otherwise} \end{cases}$$

(a) Find the value of λ. (2 marks)
Hence determine

(b) the mean, μ, and variance, σ^2, of X (8 marks)

(c) the cumulative distribution of X (1 mark)

(d) $P(|X - \mu| \le \sigma)$, giving your answer correct to 3 significant figures. (4 marks)

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8. A thin uniform rod AB of mass $2m$ and length $2a$ is free to rotate in a vertical plane about a smooth horizontal axis through a point P on the rod, where $AP = \frac{3}{2}a$. The rod is released from rest when PB makes an angle β with the horizontal.

(a) Show that in the ensuing motion,

$$3a\left(\frac{d\theta}{dt}\right)^2 = 4g(\sin\theta - \sin\beta), \quad \text{(8 marks)}$$

where θ is the angle the rod makes with the horizontal at time t.

(b) When $\beta = \frac{\pi}{3}$, find the magnitude of the reaction on the axis of rotation when $\theta = \frac{\pi}{3}$.

(7 marks)

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1. Forces $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_n$ act at the points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ respectively. State the conditions under which the system reduces to a couple. **(2 marks)**

Three forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ act at the points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ respectively where

$$\mathbf{F}_1 = (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\text{N}, \quad \mathbf{r}_1 = (2\mathbf{i} - 3\mathbf{j} - \mathbf{k})\text{m}$$
$$\mathbf{F}_2 = (4\mathbf{i} - \mathbf{j} + \mathbf{k})\text{N}, \quad \mathbf{r}_2 = (3\mathbf{i} - \mathbf{j} - \mathbf{k})\text{m}$$
$$\mathbf{F}_3 = (-3\mathbf{i} - \mathbf{j})\text{N}, \quad \mathbf{r}_3 = (\mathbf{i} - \mathbf{j} - 2\mathbf{k})\text{m}$$

(a) Find the moment of $\mathbf{F}_1$ about the origin. **(2 marks)**

(b) Show that the system is equivalent to a couple $\mathbf{G}$ of magnitude $\sqrt{148}$Nm. **(6 marks)**

2. A particle P describes the curve $r = ke^{\theta}$ where k is a positive constant in such a way that the radial component of its acceleration is zero.

(a) Show that $r^2\dot{\theta}$ = constant. **(10 marks)**

(b) Find the transverse component of the acceleration in terms of r and θ. **(2 marks)**

3. A smooth sphere S moving with velocity $2\lambda(2\mathbf{i} + \mathbf{j})\text{ms}^{-1}$, where $\lambda > 0$, collides obliquely with an identical sphere T which is at rest. After impact, sphere T starts to move in the direction of the vector $(3\mathbf{i} + \mathbf{j})$. The coefficient of restitution between the two spheres is $\frac{1}{2}$.

(a) State the direction of the impulse exerted by sphere S on sphere T. **(1 mark)**

(b) Show that if θ is the inclination of the direction of motion of sphere S to the line of impulses then $\cos\theta = \frac{2}{\sqrt{5}}$. **(4 marks)**

(c) Find the velocity of sphere T immediately after impact. **(6 marks)**

4. A model boat of mass 10kg is put in water and its engines turned on. Given that the boat experiences a resistive force of kv^2 newtons, where $v(t)$ is the instantaneous velocity of the boat in ms^{-1} at time t seconds and that the engines exert a force of 200 newtons to propel the boat forward in a straight course,

(a) Show that the equation of motion of the boat can be written in the form

$$\frac{dv}{dt} = 1 - \frac{v^2}{10^2}. \quad \textbf{(2 marks)}$$

(b) Determine the velocity $v(t)$ as a function of t. **(4 marks)**

(c) Find the terminal velocity of the boat. **(3 marks)**

(d) Show that if $x(t)$ is the distance covered by the boat, then as $t \to \infty$, $x(t) \to at + b$, and find the values of the constants a and b. **(2 marks)**

5. (i) A particle of mass 2kg is oscillating with simple harmonic motion of period π seconds. The particle passes through a point A at a distance 0.5 m from the equilibrium position while travelling with speed 3 m/s.

Find

(a) the amplitude of the simple harmonic motion. **(4 marks)**

(b) the magnitude of the force acting on the particle. **(4 marks)**

(ii) A particle P moves in a resisting medium such that its displacement x from a fixed point O is given by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 17x = 0.$$

Show that P undergoes a damped harmonic motion and find the period of this motion. **(6 marks)**